TENSORS ON A VECTOR SPACE

Let V be a ~~...~~ vector space over $\mathbb{R}$.

def: A tensor Φ on V is a ~~...~~ multilinear map

$$\Phi : \underbrace{V \times \dots \times V}_{r} \longrightarrow \mathbb{R} \qquad r = \text{covariant order}$$

$$\mathcal{T}^r(V) := \{\text{all tensors of order } r \text{ on } V\}$$

$$\boxed{\begin{aligned} \mathcal{T}^0(V) &= \mathbb{R} \\ \mathcal{T}^1(V) &= V^* \end{aligned}}$$

FACT: $\forall \alpha_1, \alpha_2 \in \mathbb{R} \quad \forall \Phi_1, \Phi_2 \in \mathcal{T}^r(V)$

we can define

$$(\alpha_1 \Phi_1 + \alpha_2 \Phi_2)(v_1, \dots, v_n) := \alpha_1 \Phi_1(v_1, \dots, v_n) + \alpha_2 \Phi_2(v_1 \dots v_n)$$

Thm/Ex: $\mathcal{T}^r(V)$ is a vector space of dim n^r on V

let $\{e_1, \dots, e_n\} \in V$ basis of V, $\phi \in \mathcal{T}^r$ is determined by its value on e_i's

we write:

$v_i = \sum \alpha_i^j \cdot e_j \quad i = 1, \dots, r$, we get

$$\Phi(v_1, \dots, v_r) = \sum_{j_1, \dots, j_r} \alpha_{i_1}^{j_1} \dots \alpha_{i_r}^{j_r} \underbrace{\Phi(e_{j_1}, \dots, e_{j_r})}$$

by bilinearity

components of Φ in the basis $e_1, \dots, e_n$

FACT 2: Any linear map $F_* : V \longrightarrow W$ induces

$$F^* : \mathcal{T}^r(W) \longrightarrow \mathcal{T}^r(V)$$

$$\Phi(v_1, \dots, v_r) \longmapsto \Phi(F_*(v_1), \dots, F_*(v_r))$$

Symmetrizing / alternating transformations

<u>def</u>: $\phi \in \mathcal{T}^r(V)$ is <u>symmetric</u> if $\forall \; 1 \leq i, j \leq r$

we have: $\Phi(v_1, \dots, v_i \dots v_j \dots v_r) = \Phi(v_1 \dots v_j \dots v_i \dots v_r)$

ϕ is <u>skew-symmetric</u> / <u>alternating</u> if:

$$\Phi(v_1, \dots, v_i \dots v_j \dots v_r) = - \Phi(v_1 \dots v_j \dots v_i \dots v_r)$$

Equivalently, ~~The Permutation~~ $\mathfrak{S}(r) = \{\text{permutations on } 1, \dots, r\}$

$\phi \in \mathcal{T}^r(V)$ <u>symmetric</u> if $\phi(v_1, \dots, v_r) = \Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)})$

<u>skew- "</u> $\phi(v_1, \dots, v_r) = \underbrace{sg(\sigma)} \Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)})$

$\forall \sigma \in \mathfrak{S}(r) \; \forall v_1, \dots, v_r \in V$

<u>def</u>: The <u>symmetrizing map</u>:

$$\mathcal{S}: \mathcal{T}^r(V) \longrightarrow \mathcal{T}^r(V)$$

$$\Phi(v_1, \dots, v_r) \longmapsto \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}(r)} \Phi(v_{\sigma(1)}; \dots v_{\sigma(r)})$$

The <u>alternating map</u>:

$$\mathcal{A}: \mathcal{T}^r(V) \longrightarrow \mathcal{T}^r(V)$$

$$\Phi(v_1, \dots, v_r) \longmapsto \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}(r)} \operatorname{sgn}(\sigma) \cdot \Phi(v_{\sigma(1)}, \dots, v_{\sigma(r)})$$

Properties :

(1) Both are linear on $\mathcal{T}^r(V)$

(2) Both are projections :

$$\mathcal{A}^2 = \mathcal{A} \qquad \mathcal{S}^2 = \mathcal{S}$$

(3) $\mathcal{A}(\mathcal{T}^r(V)) = \left\{ \begin{array}{c} \text{all skew-symmetric tensors} \\ \text{on } V \text{ of order } r \end{array} \right\} =: \Lambda^r(V)$

$\mathcal{S}(\mathcal{T}^r(V)) = \left\{ \begin{array}{c} \text{all symmetric tensors} \\ \text{on } V \text{ of order } r \end{array} \right\} =: \Sigma^r(V)$

(4) Φ alternating $\Leftrightarrow$ $\mathcal{A}\Phi = \Phi$

Φ symmetric $\Leftrightarrow$ $\mathcal{S}\Phi = \Phi$

(5) if $F_* : V \longrightarrow W$ linear then $\mathcal{A}$ and $\mathcal{S}$ commute

wt $F^*: \mathcal{T}^r(W) \longrightarrow \mathcal{T}^r(V)$

Multiplication of tensors

def: Let $\varphi \in \mathcal{T}^r(V)$ tensors.
$\psi \in \mathcal{T}^s(V)$

The product $\varphi \otimes \psi (v_1, \dots, v_r, v_{r+1}, \dots, v_{r+s}) =$

$$= \varphi(v_1, \dots, v_r) \cdot \psi(v_{r+1}, \dots, v_{r+s})$$

$$\mathcal{T}^r(V) \times \mathcal{T}^s(V) \longrightarrow \mathcal{T}^{r+s}(V)$$

$$(\varphi, \psi) \longmapsto \varphi \otimes \psi$$

Theorem :

$\mathcal{T}^r(U) \times \mathcal{T}^s(W) \longrightarrow \mathcal{T}^{r+s}(U)$ is

bilinear & associative.

If $\omega_1, \dots, \omega_n$ is a basis of $V^* = \mathcal{T}^1(V)$

then $\{\omega_{i_1} \otimes \dots \otimes \omega_{i_r}\}_{1 \le i_1, \dots, i_r \le n}$ is a basis of $\mathcal{T}^r(V)$.

Finally, if $F_* : W \longrightarrow V$ is linear $\Rightarrow F^*(\varphi \otimes \psi) =$

$= F^*(\varphi) \otimes F^*(\psi)$.

$$\mathcal{T}(U) := \mathcal{T}^0(V) \oplus \mathcal{T}^1(V) \oplus \dots \oplus \mathcal{T}^r(U) \oplus \dots$$

$\uparrow$

tensor algebra over V

Thm : $\mathcal{T}(V)$ is an associative algebra wt unit over $\mathbb{R}$

it is generated by $\mathcal{T}^0(V)$ and $\mathcal{T}^1(V) = U^*$

Any linear mapping $F_* : W \longrightarrow V$ of vector spaces

induces $F^* : \mathcal{T}(U) \longrightarrow \mathcal{T}(W)$ homomorphism of algebras

which coincides wt $\begin{cases} \text{id on } \mathcal{T}^0(V) \\ F^* \text{ on } \mathcal{T}^1(U) \end{cases}$

Exterior algebra

Similarly we can define

$$\Lambda V := \Lambda^0 V \oplus \Lambda^1 V \oplus \dots \oplus \Lambda^r V \oplus \dots$$

$$\subset \mathcal{T}^0 V \oplus \mathcal{T}^1 V \oplus \dots = \mathcal{T}(V)$$

<u>Notice</u>:

(1) $\Lambda^0 V \equiv \mathcal{T}^0 V \equiv \mathbb{R}$

(2) $\Lambda^1 V = \mathcal{T}^1 V = V^*$

(3) $\Lambda^r V \subsetneq \mathcal{T}^r V$ when $r > 1$ (ex.)

(4) $\Lambda^r V = 0$ when $r > \dim V$ (ex.)

Also, notice that if

$\varphi \in \Lambda^r V$ & $\psi \in \Lambda^s V$:

$\varphi \otimes \psi \in \mathcal{T}^{r+s}(V)$ but it might not be $\in \Lambda^{r+s} V$!

We want to define a <u>multiplication</u> on ΛV and make it a commutative algebra

<u>def</u>: (Wedge product)

$$\begin{aligned} \Lambda^r V \times \Lambda^s V &\longrightarrow \Lambda^{r+s} V \\ (\varphi, \psi) &\longmapsto \frac{(r+s)!}{r!\, s!} \mathcal{A}(\varphi \otimes \psi) \end{aligned}$$

$$\varphi \wedge \psi := \frac{(r+s)!}{r!\, s!} \mathcal{A}(\varphi \otimes \psi)$$

<u>Lemma</u> 1: $\Lambda^r(V) \times \Lambda^s(V) \longrightarrow \Lambda^{r+s} V$ is bilinear & associative

<u>proof</u>:

- <u>bilinear</u> : composition of bilinear & linear mappings
- <u>associative</u> :

it follows from the fact that :

$$\boxed{\begin{aligned} A(\varphi \otimes \psi \otimes \theta) &= A\big(A(\varphi \otimes \psi) \otimes \theta\big) \\ &= A\big(\varphi \otimes A(\psi \otimes \theta)\big) \end{aligned}}$$

↳ (exercise !)

$\boxtimes$

we have : $\varphi \in \Lambda^r V$
$\psi \in \Lambda^s V$
$\theta \in \Lambda^t V$

$$\varphi \wedge \psi = \frac{(r+s)!}{r!\, s!} A(\varphi \otimes \psi)$$

$$(\varphi \wedge \psi) \wedge \theta = \frac{(r+s+t)!}{(r+s)!\, t!} A\big((\varphi \wedge \psi) \otimes \theta\big)$$

~~To complete~~ <u>In general</u>, let $\varphi_i \in \Lambda^{r_i} V$

$$\varphi_1 \wedge \dots \wedge \varphi_k = \frac{(r_1 + r_2 + \dots + r_k)!}{r_1! \dots r_k!} A(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_k)$$

So $\Lambda(V) := \Lambda^0 V \oplus \dots \oplus \Lambda^r V \oplus \dots$ wt $\wedge$ product is an associative algebra over $\mathbb{R} = \Lambda^0 V$

ΛV called the <u>exterior algebra or Grassman algebra</u> over V

<u>lemma</u>:

If $\varphi \in \Lambda^r V$ and $\psi \in \Lambda^s V$ then

$$\varphi \wedge \psi = (-1)^{r \cdot s} \, \psi \wedge \varphi$$

<u>proof</u>:

We show $\mathcal{A}(\varphi \otimes \psi) = (-1)^{rs} \mathcal{A}(\psi \otimes \varphi) =$

$$= \frac{1}{(r+s)!} \sum_{\sigma} \operatorname{sgn}(\sigma) \varphi(v_{\sigma(1)} \ldots v_{\sigma(r)}) \psi(v_{\sigma(r+1)} \ldots v_{\sigma(r+s)})$$

$$= \frac{1}{(r+s)!} \sum_{\sigma} \operatorname{sgn}(\sigma) \psi(v_{\sigma(r+1)} \ldots v_{\sigma(r+s)}) \varphi(v_{\sigma(1)} \ldots v_{\sigma(r)})$$

τ: permutation $\begin{pmatrix} 1 \ldots s & s+1 \ldots r+s \\ r+1 \ldots r+s & 1 \ldots r \end{pmatrix}$

$$= \frac{1}{(r+s)!} \sum_{\sigma} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \, \psi(v_{\sigma\tau(1)} \ldots v_{\sigma\tau(s)}) \cdot \varphi(v_{\sigma\tau(s+1)} \ldots v_{\sigma\tau(r+s)})$$

$$= \operatorname{sgn}(\tau) \, \mathcal{A}(\psi \otimes \varphi)(v_1, \ldots, v_{r+s})$$

$$= (-1)^{rs} \cdot \mathcal{A}(\psi \otimes \varphi)(v_1, \ldots, v_{r+s})$$

$\boxtimes$

Thm :

If $r > n = \dim V \Rightarrow \Lambda^r V = \{0\}$

For $0 \le r \le n$ $\quad \dim \Lambda^r V = \binom{n}{r}$

Let $\omega^1, \dots, \omega^n$ be a basis of $\Lambda^1(V)$

Then the set $\{ \omega^{i_1} \wedge \dots \wedge \omega^{i_r} \mid 1 \le i_1 < \dots < i_r \le n \}$

is a basis of $\Lambda^r V$ and $\dim \Lambda V = 2^n$

proof :

Let $e_1, \dots, e_n$ be any basis of V. If $\varphi \in \Lambda^r V$

is alternating covariant tensor wt $r > \dim V$

$\Rightarrow$ on any basis set $\varphi(e_{i_1} \dots e_{i_r}) = 0$

Indeed some e_k must be repeated, interchanging it changes the sign of φ but leaves φ unchanged

$\Rightarrow \varphi = 0$ & $\Lambda^r V = 0$

Now suppose $0 \le r \le n$ and that $\omega_1, \dots, \omega_n$ is the basis of $V^* = \Lambda^1 V$ dual to $e_1, \dots e_n$

$\mathcal{A} : \mathcal{T}^r V \twoheadrightarrow \Lambda^r V$ surjective

$\{ \omega^{i_1} \otimes \dots \otimes \omega^{i_r} \} \mapsto \mathcal{A} \{ \omega^{i_1} \otimes \dots \otimes \omega^{i_r} \}$ spans $\Lambda^r V$

basis of $\mathcal{T}^r V$

$r! \, \mathcal{A}(\omega^{i_1} \otimes \dots \otimes \omega^{i_r}) = \omega^{i_1} \wedge \dots \wedge \omega^{i_r}$

Permuting $i_1, \dots, i_r$ leaves $\omega^{i_1} \wedge \dots \wedge \omega^{i_r}$ unchanged except for sign

$\Rightarrow \binom{n}{r}$ elements of the form $\omega^{i_1} \wedge \dots \wedge \omega^{i_r}$ wt $1 \leq i_1 < \dots < i_r \leq n$ span $\Lambda^r V$

- indipendent : assume :

$$\sum_{i_1 < \dots < i_r} \alpha_{i_1 \dots i_r} \omega^{i_1} \wedge \dots \wedge \omega^{i_r} = 0$$

$$\Rightarrow \Big(\sum_{i_1 < \dots < i_r} \alpha_{i_1 \dots i_r} \omega^{i_1} \wedge \dots \wedge \omega^{i_r} \Big)(e_{k_1}, \dots, e_{k_r}) = 0$$

$$\Rightarrow \quad \alpha_{i_1 \dots i_r} = 0$$

Corollary + $\omega^i(e_k) = \delta_{ik}$

$$\dim \Lambda V = \sum_{r=0}^{n} \dim \Lambda^r V = \sum_{r=0}^{n} \binom{n}{r} = 2^n . \quad \boxtimes$$

Thm Let V and W finite dimensional

$F_* : W \longrightarrow V$ linear map.

$\Rightarrow F^* : \Lambda V \longrightarrow \Lambda W$ is a homo. of exterior algebras.

proof: $A \circ F^* = F^* \circ A$ + everything we did $\boxtimes$

ORIENTED VECTOR SPACE

V vector space

$\{e_1, \dots, e_n\}$ bases have the same orientation if
$\{f_1, \dots, f_n\}$

$$\det(\alpha_i^j) > 0 \quad \text{where} \quad f_i = \sum_{j=1}^{n} \alpha_i^j e_i$$

Rmk: This is an equivalente class on {all bases}
& only 2 equivalence classes

def: An oriented vector space is a vector space plus one equivalente classes of bases: all those bases wt the same orientation as a chosen one (~~with the~~ The oriented or positively oriented bases).

$$\dim \Lambda^n V = \binom{n}{n} = 1 \Rightarrow \text{every non zero elmt is a basis.}$$

lemma: Let $\Omega \neq 0$ be an alternating covariant tensor on V of order $n = \dim V$
Let $e_1, \dots, e_n$ be a basis of V. Then for any set of vectors $v_1, \dots, v_n$ wt $v_i = \sum \gamma_i^j e_j$ we have

$$\Omega(v_1, \dots, v_n) = \det(\gamma_j^i)\, \Omega(e_1, \dots, e_n).$$

Explanation: up to a nonvanishing scalar Ω is the determinant of the components of its variables.